THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5540 Advanced Geometry 2016-2017
Assignment 2
Due Date: 16 Mar, 2017

Recall the axioms of incidence, betweenness and congruence for line segments:

**I1**. For any distinct points $A$, $B$, there exists a unique line $l_{AB}$ containing $A$, $B$.

**I2**. Every line contains at least two points.

**I3**. There exist three noncollinear points.

**B1**. If a point $B$ is between two points $A$ and $C$ (written as $A * B * C$), then $A$, $B$ and $C$ are distinct points on a line, and also $C * B * A$.

**B2**. For any two distinct points $A$ and $B$, there exists a point $C$ such that $A * B * C$.

**B3**. Given three distinct points on a line, one and only one of them is between the other two.

**B4**. Let $A$, $B$ and $C$ be three noncollinear points, and let $l$ be a line not containing any of $A$, $B$ and $C$. If $l$ contains a point $D$ lying between $A$ and $B$, then it must also contain either a point lying between $A$ and $C$ or a point lying between $B$ and $C$, but not both.

**C1**. Given a line segment $AB$, and given a ray $r$ originating at a point $C$, there exists a unique point $D$ on the ray $r$ such that $AB \cong CD$.

**C2**. If $AB \cong CD$ and $AB \cong EF$, then $CD \cong EF$. Every line segment is congruent to itself.

**C3**. Given three points $A$, $B$ and $C$ on a line satisfying $A * B * C$, and three further points $D$, $E$ and $F$ on a line satisfying $D * E * F$, if $AB \cong DE$ and $BC \cong EF$, then $AC \cong DF$.

1. Let $A$, $B$, $C$ and $D$ be four points. Using the axioms of incidence and betweenness and the line separation propetry, prove that:

   (a) if $A * B * C$ and $B * C * D$, then $A * B * D$ and $A * C * D$;

   (b) if $A * B * D$ and $B * C * D$, then $A * B * C$ and $A * C * D$.

2. Using the axioms of incidence and betweenness, prove that every line has infinitely many distinct points.

3. Using the axioms of incidence and betweenness, prove that for any two distinct points $A$ and $B$, there exists a point $C$ such that $A * C * B$.

   (Hint: Use (**B2**) and (**B4**) to construct a line that will be forced to meet the line segment AB but not contain $A$ or $B$.)

   (Remark: Therefore, $A$ and $B$ lie on opposite side of the line constructed by the hint.)

4. Show that the interior of a triangle is nonempty.

5. Given two distinct points $O$ and $A$, we define the *circle* with *center* $O$ and *radius* $OA$ to be the set $\Gamma$ of all points $B$ such that $OA \cong OB$.

   (a) Show that any line through $O$ meets the circle in exactly two points.

   (b) Show that a circle contains infinitely many points.

6. Consider $\mathbb{Q}^2$, with lines and notion of betweenness defined in the lecture. The standard distance function $d : \mathbb{Q}^2 \times \mathbb{Q}^2 \to [0, \infty)$ is given by:

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

   We also define that the line segments $AB$ and $CD$ are congruent if $d(A, B) = d(C, D)$.

   Show that the axiom (**C1**) does not hold in the model of geometry.

7. Consider $\mathbb{R}^2$, with lines and betweenness defined in the lecture. Define a distance function $d$ : $\mathbb{R}^2 \times \mathbb{R}^2 \to [0, \infty)$ which is not the standard one as the following:

$$d((x_1, y_1), (x_2, y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

   We also define that the line segments $AB$ and $CD$ are congruent if $d(A, B) = d(C, D)$.

   Prove that the axioms (**C1**), (**C2**) and (**C3**) holds.